# 5. On Stationary Optimal Stocks in Optimal Growth Theory: Existence and Uniqueness Results

Tapan Mitra
Department of Economics, Cornell University, Ithaca, NY, USA

Kazuo Nishimura
Institute of Economic Research, Kyoto University, Japan[1]

## 5.1 Introduction

The concept of a non-trivial stationary optimal stock (SOS) plays a central role in the theory of optimal intertemporal allocation over an infinite horizon. While the optimal policy correspondence describes fully optimal behavior in such models, it is quite difficult to compute it accurately, and it can be solved in explicit form in only a very few highly specialized examples.

However, if non-stationary optimal programs, after a period of transition, are close to a certain stationary program (and the transition period is not very long), then their behavior can be approximately described by the stationary optimal program. Thus, even though it is only by accident that an economy has exactly a stationary optimal stock as its initial stock, a study of the existence, uniqueness and (local and global) stability of stationary optimal programs is of considerable significance.

Furthermore, if one is interested in comparative dynamics in this framework, one observes that it might be very difficult to get definitive results for policy purposes by varying a parameter and seeing the effect of it on the entire optimal policy correspondence. On the other hand, if the stationary optimal program is at least locally stable, then one can often predict the change in the stationary optimal program following a "small" change in a parameter, and this can enable one to conduct local comparative dynamics exercises in this framework.

In this essay, we present the basic results on the existence and uniqueness of (non-trivial) stationary optimal programs. A comprehensive account of the

[1] Our intellectual debt to William Brock and Lionel McKenzie, for our understanding of the subject matter of this survey, should be quite obvious. In writing this survey, we have relied heavily on our collaborative research with Jess Benhabib, Swapan Dasgupta, and Ali Khan.

stability (or turnpike) property of stationary optimal programs is already available in McKenzie (1986), and we refer the reader to his definitive study of this topic.

The existence of a stationary optimal stock (briefly, SOS) in multi-sector optimal growth models has been shown by Sutherland (1970) Hansen and Koopmans (1972), Peleg and Ryder (1974), Cass and Shell (1976), Flynn (1980), McKenzie (1982, 1986) and Khan and Mitra (1986), among others. We follow very closely the approach in Khan and Mitra (1986).

The demonstration of existence typically consists of three separate steps. First, a fixed point argument is used to show the existence of what we call in the sequel, a discounted golden-rule stock. Second, a separation argument in the form of the Kuhn-Tucker theorem is used to provide a "price-support" to the discounted golden-rule stock. Finally, a computation based on the price support property is used to show that the discounted golden-rule stock is optimal among all programs starting from that stock.

This approach, relying on duality theory (in the second and third steps), is followed by Peleg and Ryder (1974), Cass and Shell (1976), Flynn (1980), McKenzie (1982, 1986). An exception to this is Sutherland (1970) who relies on methods of dynamic programming and is able to avoid supporting prices and the Kuhn-Tucker theorem. However, Sutherland does not establish the existence of a *non-trivial* SOS, and as noted by Peleg and Ryder (1974), the null stock is always a SOS in a set-up which allows for the possibility of inaction, and does not allow production of positive outputs from zero inputs.

Khan and Mitra (1986) use a purely primal approach to the existence of a non-trivial SOS, and by a simple computation based on Jensen's inequality, establish that a discounted golden-rule stock is always a SOS. Thus, once the fixed point argument (the first step in the three-step argument indicated above) ensures the existence of a discounted golden-rule stock, the existence of a stationary optimal stock is also assured. This primal approach does not suffer from the shortcoming noted in the dynamic programming method, for it is simple to identify a condition on the economy (known as $\delta - normality$) which ensures that the discounted golden-rule stock (and therefore the corresponding stationary optimal program) is non-trivial.

The existence of a discounted golden-rule stock therefore emerges as a key concept of this subject. The idea is to approach an infinite-horizon optimization problem by solving an appropriate two-period optimization problem.

A direct payoff of the primal approach of Khan and Mitra (1986) is that an assumption frequently used in this literature (known as $\delta - productivity$) can be dispensed with, since its role is simply to ensure that Slater's condition holds when one invokes the Kuhn-Tucker theorem (in the second step of the three-step argument).

Following Khan and Mitra (1986), we also use a purely primal approach to show that a SOS, $k$, is always a discounted golden-rule stock, provided $(k, k)$ is in the interior of the technology set. This result is proved by McKenzie (1986), relying on duality methods. Again, the proof involves three steps. First, a sequence of prices is found to support the stationary optimal program, following the approach of Weitzman (1970). Second, by an argument due to Sutherland (1967), a "quasi-stationary" price support is obtained from the above sequence of supporting prices. Third, this (quasi-stationary) price support property is used to show that the SOS is a discounted golden-rule stock. In dispensing with support prices, we provide a direct and short proof. We also present an example to show that the result fails when $(k, k)$ is not in the interior of the technology set.

In general, when future utilities are discounted (as we are assuming in our framework throughout), there can be multiple (non-trivial) stationary optimal stocks (even when the utility function of the economy is strictly concave, unlike in the undiscounted case). Examples of economies with more than one non-trivial stationary optimal stock were given by Kurz (1968), Liviatan and Samuelson (1969) and Sutherland (1970). However, for some classes of models, one can provide sufficient conditions under which there can be only one non-trivial SOS.

We present two distinct approaches to the uniqueness issue. First, in an economy in which production is described by a simple linear model involving no joint production, and the utility (derived from consumption alone) satisfies a normality assumption, we show that there is exactly one non-trivial stationary optimal stock, using the methods of convex analysis (and, in particular, duality theory). We also provide an example where the normality assumption is violated and there are multiple non-trivial stationary optimal stocks. These results illustrate the somewhat more general investigations along these lines presented in Brock (1973) and Brock and Burmeister (1976).

Second, using the methods of differential topology, and relying on assumptions on the Jacobian obtained from the Ramsey-Euler equations (which hold for an interior stationary optimal stock in a model in which the utility function is twice continuously differentiable in the interior of the technology set), one can view the uniqueness result for interior stationary optimal stocks in the discounted case as following from the uniqueness result in the undiscounted case. Our approach follows Benhabib and Nishimura (1979), which generalizes a result along these lines by Brock (1973).

## 5.2 Preliminaries

### 5.2.1 Notation

Let $\mathbb{N}$ be the set of non-negative integers $\{0, 1, 2, ...\}$, and let $n$-dimensional Euclidean space be denoted by $\mathbb{R}^n$, where $\|x\|$ denotes the Euclidean norm of any element $x$ in $\mathbb{R}^n$. For any $x, y$ in $\mathbb{R}^n$, we shall write $x \gg y (x \geq y)$ to denote $x_i > y_i (x_i \geq y_i)$ for all coordinates $i = 1, ..., n$; and $x > y$ to denote $x \geq y$ and $x \neq y$. For any set, $S$, the set of all subsets of $S$ will be denoted by $\mathcal{B}(S)$ and hence we shall write $\phi : X \to \mathcal{B}(Y)$ for any correspondence (set-valued map) $\phi$

with domain $X$ and range $\mathcal{B}(Y)$. Finally, let $e$ denote an element of $\mathbb{R}^n_+$, all of whose coordinates are unity.

### 5.2.2 The Model

The framework is described by a triplet $(\Omega, u, \delta)$, where $\Omega$, a subset of $\mathbb{R}^n_+ \times \mathbb{R}^n_+$, is a *transition possibility set*, $u : \Omega \to R$ is a *utility function* defined on this set, and $\delta$ is the *discount factor* satisfying $0 < \delta < 1$. A typical element of $\Omega$ is written as an ordered pair $(x, y)$; this means that if the current state is $x$, then it is possible to be in the state $y$ in one period.

We will be using the following assumptions:

(A.1) (i) $(0,0) \in \Omega$; (ii) $(0,y) \in \Omega$ *implies* $y = 0$.

(A.2) $\Omega$ *is* (i) *closed, and* (ii) *convex.*

(A.3) *There is* $\xi$ *such that* "$(x,y) \in \Omega$ *and* $\|x\| \geq \xi$" *implies* "$\|y\| < \|x\|$".

(A.4) *If* $(x,y) \in \Omega$ *and* $x' \geq x$, $0 \leq y' \leq y$, then (i) $(x', y') \in \Omega$ *and* (ii) $u(x', y') \geq u(x, y)$.

(A.5) $u$ *is* (i) *upper semicontinuous and* (ii) *concave* on $\Omega$.

(A.6) *There is* $\zeta$ *such that* $(x,y) \in \Omega$ *implies* $u(x,y) \geq \zeta$.

A *program* from $y \in \mathbb{R}^n_+$ is a sequence $\{y(t)\}_0^\infty$ such that $y(0) = y$, and $(y(t),\, y(t+1)) \in \Omega$ for $t \geq 0$.

A program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}^n_+$ is an *optimal program* if

$$\sum_{t=0}^{\infty} \delta^t u(y'(t), y'(t+1)) \leq \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1))$$

for every program $\{y'(t)\}_0^\infty$ from $y$.

An optimal program $\{y(t)\}_0^\infty$ from $y \in \mathbb{R}^n_+$ is a *stationary optimal program* if $y(t) = y(t+1)$ for $t \geq 0$. A *stationary optimal stock* is an element $y \in \mathbb{R}^n_+$, such that $\{y\}_0^\infty$ is a stationary optimal program. It is *non-trivial* if $u(y,y) > u(0,0)$.

A *discounted golden-rule stock* $k$ is an element of $\mathbb{R}^n_+$ such that

(i) $(k,k) \in \Omega$

(ii) $u(k,k) \geq u(x,y)$ for all $(x,y) \in \Omega$ such that $\delta y - x \geq (\delta - 1)k$.

It is non-trivial if $u(k,k) > u(0,0)$.

### 5.2.3 Existence of Optimal Programs and the Principle of Optimality

The following "boundedness properties" of our model are well-known.

(R.1) *Under assumptions* (A.3) *and* (A.4)(i) ,

(i) *If* $(x,y) \in \Omega$, *then* $\|y\| \leq \max[\xi,\, \|x\|]$.

(ii) *If* $\{y(t)\}_0^\infty$ *is a program from* $y \in \mathbb{R}^n_+$, *then* $\|y(t)\| \leq \max[\xi,\, \|y\|]$ *for* $t \geq 0$.

The existence of an optimal program in this framework is also a standard result.

(R.2) *Under assumptions* (A.1), (A.2), (A.3), (A.4) (i), (A.5) (i) *and* (A.6), *if* $y \in \mathbb{R}^n_+$, *there exists an optimal program from* $y$.

Given (R.2), there is an optimal program $\{y^*(t)\}_0^\infty$ from each $y \in \mathbb{R}^n_+$. We define

$$V(y) = \sum_{t=0}^{\infty} \delta^t u(y^*(t),\, y^*(t+1))$$

$V$ is known as the *value function.*

The following result is standard and is known as the "principle of optimality".

(R.3) If $\{y(t)\}_0^\infty$ is an optimal program from $y$, then

$$V(y) = \sum_{t=0}^{N} \delta^t u(y(t), y(t+1)) + \delta^{N+1} V(y(N+1)) \quad \text{for} \quad N \geq 0.$$

## 5.3 Equivalence of Discounted Golden-Rule and Stationary Optimal Stocks

A stationary optimal stock constitutes a solution to an infinite horizon problem. It is a stock such that, if one starts from it, then among all programs starting from it (whether stationary or not), the program which remains stationary at the initial stock is optimal. Yet the stationary nature of the solution makes it plausible to conjecture that one might be able to find it by solving a finite-horizon problem. The equivalence of a discounted golden-rule stock and a stationary optimal stock shows that this is indeed the case, as the discounted golden-rule might be seen as the solution to a problem involving two periods.

Our approach to this equivalence result follows Khan and Mitra (1986). It is "primal" in that it makes no use of supporting prices, unlike most treatments of it in the literature, which rely on duality theory.

**Theorem 5.3.1.** *Every discounted golden-rule stock $k$ is a stationary optimal stock.*

*Proof.* Let $\{y(t)\}_0^\infty$ be any program from $k$. We shall show that it does not give a higher discounted utility sum than the stationary program $\{k\}_0^\infty$.

Let $x(T) = \sum_{t=0}^{T-1}(1-\delta)\delta^t y(t)/(1-\delta^T)$ and $z(T) = \sum_{t=0}^{T-1}(1-\delta)\delta^t y(t+1)/(1-\delta^T)$. Given convexity of $\Omega$, certainly $(x(T), y(T)) \in \Omega$ for all $T \geq 1$. We know that $y(t)$ is bounded independently of $t$. Hence $(\bar{x}, \bar{z}) = \lim_{T\to\infty}(x(T), z(T))$ is well-defined and is an element of $\Omega$.

Now, by the fact that $0 < \delta < 1$, Jensen's inequality yields $u(\bar{x}, \bar{z}) \geq \sum_{t=0}^{\infty}(1-\delta)\delta^t u(y(t),\ y(t+1))$. But $(\bar{x} - \delta\bar{z}) = (1-\delta)[\sum_{t=0}^{\infty}\delta^t y(t) - \sum_{t=0}^{\infty}\delta^{t+1} y(t+1)] = (1-\delta)k$. Since $(k, k)$ is a discounted golden-rule stock, certainly $u(k, k) \geq u(\bar{x}, \bar{z})$, which implies:

$$\sum_{t=0}^{\infty} \delta^t u(k,k) \geq \sum_{t=0}^{\infty} \delta^t u(\bar{x},\bar{z}) = u(\bar{x},\bar{z})/(1-\delta) \geq \sum_{t=0}^{\infty} \delta^t u(y(t),y(t+1))$$

We can now state a converse to Theorem 5.3.1.

**Theorem 5.3.2.** *Every stationary optimal stock $k$ such that $(k,k) \in$ interior $\Omega$, is a discounted golden-rule stock.*

*Proof.* Suppose not; then there exists $(x,y) \in \Omega$ such that $\delta y - x \geq \delta k - k$ and $u(x,y) > u(k,k)$. Since $u$ is non-decreasing in the first component, we can assume without any loss of generality that $x = (1-\delta)k + \delta y$. Let $\gamma \equiv u(x,y) - u(k,k) > 0$.

Using $(x,y)$, we shall now construct a program $\{y'(t)\}_0^\infty$ starting from $k$ that gives more discounted sum of utilities than the stationary optimal program $\{k\}_0^\infty$. This furnishes us the required contradiction. Towards this end, for a value of $N$ to be determined later, let:

$$z(q) = (1-\delta^q)k + \delta^q x \quad \text{for all } q = 0,...,N$$

Then, we have for all $q = 1,...,N$,

$$\begin{aligned} z(q-1)) &= (1-\delta^{q-1})k + \delta^{q-1}x \\ &= (1-\delta^q)k + \delta^q y \end{aligned}$$

using the fact that $x = (1-\delta)k + \delta y$. Thus, we have:

$$(z(q), z(q-1)) = (1-\delta^q)(k,k) + \delta^q(x,y) \text{ for all } q = 1,...,N. \tag{5.1}$$

By convexity of $\Omega$, we have $(z(q), z(q-1)) \in \Omega$ for all $q = 1,...,N$. Now let $\{y'(t)\}_0^\infty$ be defined by $y'(0) = k$, $y'(t) = z(N-t+1)$, for $t = 1,...,N$; $y'(N+1) = z(0) = x$; $y'(t) = 0$ for $t \geq N+2$.

We now show that for large enough $N$, $\{y'(t)\}_0^\infty$ is a program. For this, it only remains to show that $(k, y'(1)) = (k, z(N)) \in \Omega$. But $(k,k) \in$ *interior* $\Omega$, and so there exists $\alpha > 0$ such that $(k,y) \in \Omega$ for all $y \in S_2 \equiv \{y : k - 2\alpha e << y << k + 2\alpha e\}$. Let $S_1 = \{y : k - \alpha e \leq y \leq k + \alpha e\}$. From the definition of $z(q)$, it is clear that $z(q) - \delta z(q-1) = (1-\delta)k$ for $q = 1,...,N$, which implies $(z(q) - k) = \delta(z(q-1) - k)$. Since $\delta$ is less than 1, certainly $z(q) \to k$ as $q \to \infty$ and hence there exists $N_1$ such that $z(N) \in S_1$ for all $N \geq N_1$.

Next, we can assert, using the concavity of $u$, that for all $q = 1,...,N$,

$$u(z(q), z(q-1)) \geq (1-\delta^q)u(k,k) + \delta^q u(x,y) \geq u(k,k) + \delta^q \gamma.$$

By Mangasarian (7, p. 63), it is also true that

$$\|u(k, z(N)) - u(k,k)\| \leq A \|z(N) - k\| = A\delta^{N+1} \|y - k\|,$$

where $A \equiv (u(k,k) + \hat{\beta})/\alpha, \hat{\beta} = -Min_{y \in W} u(k,y)$ and $W$ is the set of $2n$ vertices of $S_1$. Hence we have

$$\sum_{t=0}^{N+1} \delta^t[u(y'(t), y'(t+1)) - u(k,k)] \geq -A\delta^{N+1} \|y-k\| + (N+1)\delta^{N+1}\gamma.$$

On adding terms after the time period $(N+1)$, we obtain:

$$\begin{aligned} & \sum_{t=0}^{\infty} \delta^t[u(y'(t), y'(t+1)) - u(k,k)] \\ \geq \;& \delta^{N+1}((N+1)\gamma - A\|y-k\| + \{\delta u(0,0)/(1-\delta)\} - \delta V(k)). \quad (5.2) \end{aligned}$$

Let $N_2$ be a value of $N$ such that the right-hand side of (5.2) is positive, and let $N' = Max(N_1, N_2)$. Now any $\{y'(t)\}_0^\infty$ with $N \geq N'$ furnishes us with a contradiction to the fact that $\{k\}_0^\infty$ is a stationary optimal program.

A natural question arises as to whether the interiority hypothesis in Theorem 5.3.2 can be dispensed with. The following example shows this not to be the case.

**Example 1:**

Let $\Omega = \{(x,y) \in \mathbb{R}^2_+ \times \mathbb{R}^2_+ : Ay \leq x, \;\; ey \leq 3\}$, where:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$$

and $e = (1,1)$. Let $\delta = 1/2$ and $u(x,y) = ex$. It is clear that this economy satisfies all the assumptions made in Section 5.2. We shall show that $k = (1,0)$ is a stationary optimal stock. To this end, observe that $(k,k) \in \Omega$ and consider any program $\{y(t)\}_0^\infty$ starting from $k$. Since $(y(t), y(t+1)) \in \Omega$, we have $y(t) \leq (1,0)$ for all $t$. Hence,

$$\begin{aligned} \sum_{t=0}^{\infty} \delta^t u(y(t), y(t+1)) &= \sum_{t=0}^{\infty} \delta^t (ey(t)) \leq \sum_{t=0}^{\infty} \delta^t \\ &= \sum_{t=0}^{\infty} \delta^t (ek) = \sum_{t=0}^{\infty} \delta^t u(k,k). \end{aligned}$$

Now let $x' = (1,1)$, $y' = (1,2)$. Certainly $(x', y') \in \Omega$ and $\delta y' - x' = \delta k - k$. But $u(x', y') = ex' = 2 > ek = u(k,k)$ and thus $k$ is not a discounted golden-rule stock.

## 5.4 Existence of Discounted Golden-Rule and Stationary Optimal Stocks

Given the equivalence result of Section 5.3, the existence of a stationary optimal stock can be established by showing that there exists a discounted golden-rule stock. Since one can easily impose conditions on the economy to ensure that the discounted golden-rule stock obtained is non-trivial, this approach has the advantage of identifying conditions on the economy sufficient for the existence of a *non-trivial* stationary optimal stock. This advantage is not shared by the dynamic programming approach followed by Sutherland (1967), a shortcoming that was pointed out by Peleg and Ryder (1974).

**Lemma 5.4.1.** *Let $S = \{x \in \mathbb{R}^n_+ : \|x\| \leq \beta\}$ and $\phi$ and $\psi$ be mappings from $S$ into $\mathcal{B}(\mathbb{R}^n_+ \times \mathbb{R}^n_+)$ such that for $z \in S$, $\phi(z) = \{(x, y) \in \Omega : \delta y - x \geq \delta z - z\}$ and $\psi(z) = \{(x, y) \in \phi(z) : u(x, y) \geq u(x', y')$ for all $(x', y') \in \phi(z)\}$. Then, $\psi$ is a non-empty, convex-valued, and upper semicontinuous correspondence.*

*Proof.* Clearly, $S$ is a non-empty, convex, and compact set. Next, we claim that $\phi$ is a non-empty, compact-valued correspondence. For any $z \in S$, we have $(0, 0) \in \phi(z)$, and, since $\Omega$ is convex and closed, $\phi(z)$ is convex and closed. Furthermore, if $(x, y) \in \phi(z)$, then $\|x\| \leq \beta$. This implies, in turn, that if $(x, y) \in \phi(z)$, then $\|y\| \leq \beta$. Thus on defining $S' = \{(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : \|y\| \leq \beta\}$, we note that $S'$ is a non-empty, compact set, and for any $z \in S$, $\phi(z)$ is a subset of $S'$. Since $\phi(z)$ is closed for each $z \in S$, $\phi(z)$ is compact for each $z \in S$.

Since $u$ is an upper semicontinuous function on $\Omega$, and $\phi(z)$ is a non-empty, compact subset of $\Omega$, $\psi(z)$ is non-empty for each $z \in S$. It is also convex by concavity of $u$ and convexity of $\phi(z)$.

Next, we show the upper semicontinuity of $\psi$. Let $z^*$ be an arbitrary point of $S$. Consider a sequence $\{z^n\}$, with $z^n \in S$, for $n = 1, 2, 3, ...$, with $z^n \to z^*$ as $n \to \infty$. Let $(x^n, y^n) \in \psi(z^n)$, and $(x^n, y^n) \to (\hat{x}, \hat{y})$. We want to show that $(\hat{x}, \hat{y}) \in \psi(z^*)$. Since $\Omega$ is closed, $(\hat{x}, \hat{y}) \in \phi(z^*)$. Suppose $(\hat{x}, \hat{y}) \notin \psi(z^*)$. Then there is some $(x^*, y^*) \in \psi(z^*)$ and an $\varepsilon > 0$ such that $u(x^*, y^*) \geq u(\hat{x}, \hat{y}) + \varepsilon$.

Now, since $u$ is an upper semicontinuous function, $\lim_{n\to\infty} \sup u(x^n, y^n) \leq u(\hat{x}, \hat{y})$. Thus, there is $N_1$ such that for $n \geq N_1$, $u(x^n, y^n) \leq u(\hat{x}, \hat{y}) + \varepsilon/3$. Consequently, for $n \geq N_1$,

$$u(x^*, y^*) \geq u(x^n, y^n) + 2\varepsilon/3. \tag{5.3}$$

Choose $0 < \lambda < 1$ such that $(1 - \lambda)[u(0, 0) - u(x^*, y^*)] \geq -\varepsilon/3$. We claim that there is an $N_2$ such that for $n \geq N_2$, $(\lambda x^*, \lambda y^*) \in \phi(z^n)$. To see this, observe that $(0, 0) \in \Omega$ and convexity of $\Omega$ imply that $(\lambda x^*, \lambda y^*) \in \phi(\lambda z^*)$. Since $z^n \to z^*$, there is $N_2$ such that for $n > N_2$, $z^n \geq \lambda z^*$. Thus $\delta\lambda y^* - \lambda x^* \geq (\delta - 1)\lambda z^* \geq (\delta - 1)z^n$, establishing our claim.

Since $(x^n, y^n) \in \psi(z^n)$, for $n \geq N_2$,

$$\begin{aligned} u(x^n, y^n) &\geq u(\lambda x^*, \lambda y^*) \geq \lambda u(x^*, y^*) + (1 - \lambda)u(0, 0) \\ &= u(x^*, y^*) + (1 - \lambda)[u(0, 0) - u(x^*, y^*)] \\ &\geq u(x^*, y^*) - \varepsilon/3. \end{aligned}$$

Using this in (5.3) for $n \geq Max(N_1, N_2)$,

$$u(x^*, y^*) \geq u(x^n, y^n) + 2\varepsilon/3 \geq u(x^*, y^*) + \varepsilon/3,$$

a contradiction, which completes the proof.

**Theorem 5.4.1.** *There exists a discounted golden-rule stock.*

*Proof.* Define $Q : S \to \mathcal{B}(\mathbb{R}^n_+)$, where for $z \in S$, $Q(z) = \{x \in \mathbb{R}^n_+ : (x, y) \in \psi(z)\}$. We will show that this correspondence $Q$ satisfies all the requirements of Kakutani's fixed-point theorem.

Lemma 5.4.1 implies that $Q$ is a non-empty, convex-valued correspondence. It also implies that $Q$ is upper semicontinuous. To see this, take an arbitrary $z^* \in X$. Let $z^n \in S$, with $z^n \to z^*$ as $n \to \infty$. Let $x^n \in Q(z^n)$, and $x^n \to \hat{x}$ as $n \to \infty$. We have to show that $\hat{x} \in Q(z^*)$. Since $x^n \in Q(z^n)$, there is $y^n$ such that $(x^n, y^n) \in \psi(z^n)$. This means $(x^n, y^n) \in \phi(z^n)$, and by compactness of $\phi(z^n)$, we can pick a subsequence $(x^{n'}, y^{n'})$ tending to $(\hat{x}, \hat{y}) \in \phi(z^*)$. By the lemma, $(\hat{x}, \hat{y}) \in \psi(z^*)$ and the claim is proved.

Thus, all the conditions of Kakutani's fixed point theorem are fulfilled, and there exists $x^0 \in Q(x^0)$. This means there is some $y^0$ such that $(x^0, y^0) \in \psi(x^0)$; that is,

$$u(x^0, y^0) \geq u(x, y) \quad \text{for all} \quad (x, y) \in \phi(x^0).$$

But $(x^0, y^0) \in \phi(x^0)$ implies $x^0 \leq y^0$, and we obtain that $(x^0, x^0) \in \Omega$, and $u(x^0, x^0) \geq u(x^0, y^0) \geq u(x, y)$ for all $(x, y) \in \Omega$, with $\delta y - x \geq \delta x^0 - x^0$. Thus, by definition, $x^0$ is a discounted golden-rule stock.

An economy $(\Omega, u, \delta)$ is called $\delta -$ *normal* if there exists $(\bar{x}, \bar{y}) \in \Omega$ such that $\bar{x} \leq \delta\bar{y}$ and $u(\bar{x}, \bar{y}) > u(0, 0)$.

**Theorem 5.4.2.** *If the economy $(\Omega, u, \delta)$ is $\delta$-normal, there exists (i) a non-trivial discounted golden-rule stock, and (ii) a non-trivial stationary optimal stock.*

*Proof.* By Theorem 5.4.1, there is a discounted golden-rule stock, $x^0$. Given $\delta$-normality, there is $(\bar{x}, \bar{y}) \in \Omega$ such that $\delta\bar{y} - \bar{x} \geq 0 \geq \delta x^0 - x^0$, and $u(\bar{x}, \bar{y}) > u(0, 0)$. Thus, by definition of a discounted golden-rule stock, $u(x^0, x^0) \geq u(\bar{x}, \bar{y}) > u(0, 0)$, and hence $x^0$ is a non-trivial discounted golden-rule stock.

By Theorem 5.3.1, $x^0$ is a stationary optimal stock, and since we have already checked that $u(x^0, x^0) > u(0, 0)$, it is a non-trivial stationary optimal stock.

**Remark:**

An economy $(\Omega, u, \delta)$ is called $\delta -$ *productive* if there exists $(\bar{x}, \bar{y}) \in \Omega$ such that $\delta\bar{y} >> \bar{x}$. Flynn (1980) establishes a version of Theorem 5.4.2 under the additional assumption of $\delta$-productivity. This is because, after establishing the existence of a discounted golden-rule, he uses the dual approach to show that the discounted golden-rule stock is a stationary optimal stock, by providing an appropriate price-support. Then $\delta-$*productivity* ensures that Slater's condition is satisfied in the application of the Kuhn-Tucker theorem.

We show now that there exist economies satisfying the hypotheses of Theorem 5.4.2, whose technologies are not $\delta$-productive, and for which there exists a non-trivial SOS.

**Example 2:**

Let $f(x) = 2x$ for $0 \leq x \leq 1$ and $f(x) = 2 + (x-1)/2$ for $x \geq 1$. Let $\Omega = \{(x,y) \in \mathbb{R}^n_+ : 0 \leq y \leq f(x)\}$, $u(x,y) = 2f(x) - y$ and $\delta = 1/2$.

Now $(\bar{x}, \bar{y}) \equiv (1,2) \in \Omega$. Certainly $\delta\bar{y} - \bar{x} = 0$ and $u(\bar{x}, \bar{y}) = 2 > 0 = u(0,0)$. Hence the economy is $\delta$-normal. Also, for any $(x,y) \in \Omega$, $\delta y - x \leq (1/2)f(x) - x \leq 0$, since for $x \geq 1$, $f(x) \leq 2x$. Thus, there cannot exist any $(x,y) \in \Omega$ such that $x << \delta y$ and so the economy is not $\delta$-productive.

Next, we claim that $x^* = 1$ is a discounted golden-rule stock. Pick any $(x,y) \in \Omega$ such that $\delta y - x \geq (\delta - 1)x^*$. Then $y \geq 2x - 1$ and $u(x,y) \leq 2f(x) - 2x + 1$. Now

$$u(x,y) \leq 2(2x) - 2x + 1 \leq 3 \qquad \text{for} \quad 0 \leq x \leq 1$$

and

$$u(x,y) \leq 2(2 + (1/2)(x-1)) - 2x + 1 \leq 3 \qquad \text{for} \quad x \geq 1.$$

In either case, $u(x,y) \leq 3 = u(1,1) = u(x^*, x^*)$, and our claim is proved.

It should be noted that $x^* = 1$ is an SOS by Theorem 5.3.1, which is non-trivial, since $u(1,1) = 3 > 0 = u(0,0)$.

We now present an example of an economy which satisfies all the assumptions of Section 5.2, and which is $\delta -$ *productive*, but which has only a trivial SOS. This economy violates the $\delta -$ *normality* assumption, showing thereby that Theorem 5.4.2 would not be valid if the $\delta -$ *normality* hypothesis is dropped from its statement.

**Example 3:**

Let $\Omega = \{(x,y) \in \mathbb{R}^2_+ : 0 \leq y \leq 2x^{1/2}\}$, $\delta = 1/2$, and $u(x,y) = x - 2y$. For $(\hat{x}, \hat{y}) = (1/4, 1) \in \Omega$, we have $\delta\hat{y} >> \hat{x}$ and so the economy is $\delta -$ *productive*. For any program $\{k\}_0^\infty$ with $0 < k \leq 4$, we have $\sum_{t=0}^\infty \delta^t u(k,k) < 0$, and so it is dominated by the program $\{y(t)\}_0^\infty$ with $y(0) = k$ and $y(t) = 0$ for $t = 1, 2, ...$ Since there is no stationary program $\{k\}_0^\infty$ with $k > 4$, $\{0\}_0^\infty$ is the unique stationary optimal program.

## 5.5 Uniqueness of Non-trivial SOS

In this section we establish the uniqueness of non-trivial stationary optimal stocks in a framework in which the technology is described by a simple linear model (see Gale(1960)) involving no joint production, and the welfare function, describing the utility derived from consumption (alone), satisfies a *normality* condition[2]. We follow closely the approach, pioneered by Brock (1973), and

[2] Optimal programs in a similar framework, but without the normality condition, have been studied in detail by Dasgupta and Mitra (1999).

developed further in Brock and Burmeister (1976). However, we rely entirely on the methods of convex analysis, and do not make any differentiability assumptions.

### 5.5.1 Description of the Framework

We describe the production side by an $n \times n$ non-negative matrix $A = (a_{ij})$, where $i = 1, ..., n$ and $j = 1, ..., n$, and a strictly positive vector $b = (b_1, ..., b_n) >> 0$. Here, $a_{ij}$ and $b_j$ are respectively the amounts of the $i-$th good and labor which are required per unit output of the $j-$th good. The total amount of labor available for production is stationary and is normalized to 1. For each $j = 1, ..., n$, it is assumed that there is some $i = 1, ..., n$ such that $a_{ij} > 0$. Thus, each production process requires a positive amount of labor as well as a positive amount of some produced factor. Further, it is assumed that $A$ is productive; that is, there is some $\tilde{y} >> 0$ such that $\tilde{y} >> A\tilde{y}$ and $b\tilde{y} \leq 1$. This essentially excludes the economically uninteresting case of a production system which is unable to sustain some positive consumption levels for all of the desired goods. The fact that $A$ is productive ensures that $(I - A)$ is non-singular, and $(I - A)^{-1} \geq 0$. The transition possibility set for this economy is given by:

$$\Omega = \{(x, y) \in \mathbb{R}_+^{2n} : Ay \leq x \text{ and } by \leq 1\}$$

We will assume in addition to the requirements stated above that $A$ is *indecomposable*; that is, there is no non-empty proper subset $J$ of $\{1, 2, ..., n\}$ such that $a_{ij} = 0$ for $i \notin J,\ j \in J$. In this case, we have the stronger result that $(I - A)^{-1} >> 0$. It is also known that $A$ has a unique Frobenius root, $\theta$, which is positive, and a real Frobenius vector, $x^*$, which is strictly positive (and taken henceforth to be normalized so that $bx^* = 1$). Since $A$ is productive, we know that $\theta \in (0, 1)$. We make the stronger assumption that:

$$0 < \theta < \delta \tag{DF}$$

where $\delta \in (0, 1)$ is the discount factor. Since $\theta \in (0, 1)$, assumption (DF) will always be satisfied for all discount factors close to 1. But, (DF) gives an explicit lower bound for the discount factor under which the uniqueness theory, to be described below, is valid. Thus, (DF) links the level of impatience, an aspect of intertemporal preferences ($\delta$), with a measure of the productivity of the economy ($\theta$). Under (DF), we have the important result[3] that $(\delta I - A)$ is also non-singular, and:

$$(\delta I - A)^{-1} >> 0 \tag{5.4}$$

Welfare is derived from consumption, as given by a function $w : \mathbb{R}_+^n \rightarrow \mathbb{R}$, which is continuous, concave and monotone on $\mathbb{R}_+^n$. In what follows, we normalize $w(0) = 0$, and assume that $w(c) > 0$ if and only if $c >> 0$. We make

[3] All the results relating to the Frobenius theorem that are stated in this paragraph can be found in Nikaido (1968, p.102-108).

stronger assumptions on $w$ when consumption is strictly positive. Specifically, we assume that $w$ is strongly monotone and strictly concave on $\mathbb{R}^n_{++}$.

We now describe the crucial *normality* assumption on $w$. Suppose $p \in \mathbb{R}^n_{++}$ and $M \in \mathbb{R}_{++}$; consider the optimization problem described by:

$$\left.\begin{array}{cc} Maximize & w(c) \\ subject\ \ to & pc \le M \\ and & c \ge 0 \end{array}\right\} (P)$$

Clearly, under our assumptions, there is a unique solution $c(p, M)$ to the problem $(P)$.

We assume that this solution is strongly monotone in $M$. That is, if $p \in \mathbb{R}^n_{++}$, $M \in \mathbb{R}_{++}$ and $M' > M$, then:

$$c(p, M') >> c(p, M) \tag{N}$$

This is known as the *normality* assumption on $w$, since it is satisfied when all goods are normal goods (in the sense used in standard consumer behavior theory).

Given $w$, the (reduced form) utility function for our framework is defined by:

$$u(x, y) = w(x - Ay) \ \text{ for all } (x, y) \in \Omega$$

It can be checked that the economy $(\Omega, u, \delta)$ as defined above satisfies all the assumptions that were stated in Section 5.2.

If $\{y(t)\}$ is a program from $y$, we will associate with it a consumption sequence $\{c(t)\}$ given by:

$$c(t) = y(t) - Ay(t + 1) \ \ for\ all \ \ t \in \mathbb{N}$$

### 5.5.2 A Uniqueness Result Under Normality

We now proceed to investigate the nature of stationary optimal stocks in the framework described in the above subsection. To this end, we first summarize in a couple of Lemmas some basic properties of any non-trivial SOS. Then, we establish the uniqueness of non-trivial SOS under the normality assumption (N).

**Lemma 5.5.1.** *If $y$ is a non-trivial SOS, then (i) $c >> 0$, and (ii) $y >> 0$.*

*Proof.* Since $y$ is a non-trivial SOS, we have $u(y, y) > u(0, 0) = 0$. Thus, we obtain $w(c) = w(y - Ay) = u(y, y) > 0$, and by our assumption on $w$, we must have $c >> 0$.

Since $c = y - Ay = (I - A)y$, and $(I - A)$ is non-singular, with $(I - A)^{-1} >> 0$, we have $y = (I - A)^{-1}c >> 0$.

The above lemma allows us to invoke a standard result on duality theory, to provide a price support, $q$, to a non-trivial SOS, $y$; the quantity-price pair $(y, p)$ is usually referred to as a modified golden-rule.

**Lemma 5.5.2.** *If $\bar{y}$ is a non-trivial SOS, then there is $\bar{q} \in \mathbb{R}^n_+$, such that:*

$$w(\bar{c}) - \bar{q}\bar{c} \geq w(c) - \bar{q}c \quad \textit{for all } c \geq 0 \tag{5.5}$$

*and:*

$$\bar{q}(\delta\bar{y} - A\bar{y}) \geq \bar{q}(\delta y - x) \quad \textit{for all } (x, y) \in \Omega \tag{5.6}$$

*Furthermore, any $\bar{q}$ satisfying (5.5) and (5.6) and $\bar{v} \equiv \bar{q}(\delta\bar{y} - A\bar{y})$ must satisfy:*

$$\bar{q}(\delta I - A) = \bar{v}b \tag{5.7}$$

*and:*

$$(i)\ \bar{q}(\delta\bar{y} - A\bar{y}) > 0, \ \ (ii)\ \bar{q} >> 0 \tag{5.8}$$

*And, $\bar{y}$ must satisfy $b\bar{y} = 1$.*

*Proof.* The fact that there exists $\bar{q} \in \mathbb{R}^n_+$ such that (5.5) and (5.6) holds, follows from a standard application of duality theory. We proceed to verify (5.7).

Clearly, we have $\bar{v} \geq 0$, since $(0, 0) \in \Omega$. Define $Y = \{\ y \in \mathbb{R}^n_+ : by = 1\}$. Then, we have, using (5.6), for all $y \in Y$,

$$0 = \bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v} \geq \bar{q}(\delta y - Ay) - \bar{v} = \bar{q}(\delta y - Ay) - \bar{v}by \tag{5.9}$$

Thus, for all $y \in Y$, we have:

$$\bar{q}(\delta y - Ay) - \bar{v}by \ \leq 0 \tag{5.10}$$

Now, let $y$ be an arbitrary vector in $\mathbb{R}^n_+$, $y \neq 0$. Then, there is $\lambda > 0$, such that $y' \equiv \lambda y$ is in $Y$. Applying (5.10) to $y'$, we have:

$$\bar{q}(\delta y' - Ay') - \bar{v}by' \ \leq 0$$

and so $\bar{q}(\delta y - Ay) - \bar{v}by \ \leq 0$ must hold. This inequality also clearly holds for $y = 0$. So, to summarize, we have now verified that:

$$\bar{q}(\delta y - Ay) - \bar{v}by \ \leq 0 \quad \textit{for all } y \geq 0 \tag{5.11}$$

Clearly, we have $b\bar{y} \leq 1$, and so:

$$\bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v}b\bar{y} \geq \bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v} = 0 \tag{5.12}$$

Combining (5.11) and (5.12), we obtain:

$$\bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v}b\bar{y} = 0 \tag{5.13}$$

Using (5.11) and (5.13), we conclude that:

$$\bar{q}(\delta\bar{y} - A\bar{y}) - \bar{v}b\bar{y} \geq \bar{q}(\delta y - Ay) - \bar{v}by \quad \textit{for all } y \geq 0 \tag{5.14}$$

Since $\bar{y} >> 0$ by Lemma 5.5.1, (5.14) yields (5.7).

We now proceed to verify (5.8). To this end, first note that $\bar{q} \neq 0$. For if $\bar{q} = 0$, then by using (5.5), we must have $w(\bar{c}) \geq w(c)$ for all $c \geq 0$; but since $\bar{c} >> 0$, this inequality would be violated for $c = 2\bar{c}$.

We now claim that (5.8)(i) must hold. For, if it did not hold, then $v = 0$, and (5.7) would yield $\bar{q}(\delta I - A) = 0$. But, since $(\delta I - A)$ is non-singular, we must then have $\bar{q} = 0$, a contradiction.

Using (5.7) and (5.8)(i), we have $\bar{q} = \bar{v}b(\delta I - A)^{-1} >> 0$, since $(\delta I - A)^{-1} >> 0$, thereby establishing (5.8)(ii).

By the definition of $\bar{v}$ and (5.13), we have $\bar{v}b\bar{y} = \bar{v}$, so that $b\bar{y} = 1$, since $\bar{v} > 0$.

**Remark:**

We note that if $\bar{y}$ is a non-trivial SOS, then by Lemma 5.5.2, $b\bar{y} = 1$, and so non-trivial stationary optimal stocks can never be in the interior of $\Omega$ in this framework.

We now turn to the uniqueness result, illustrating the role of the normality assumption on $w$.

**Theorem 5.5.1.** *There is only one non-trivial SOS.*

*Proof.* We know that there exists a non-trivial SOS in this framework, by using Theorem 5.4.2; one can check, using (DF), that $(\delta x^*, x^*) \in \Omega$ satisfies $\delta-normality$, where $x^*$ is the Frobenius vector of $A$.

To establish uniqueness, suppose on the contrary that there are two non-trivial stationary optimal stocks, $y', y''$, with $y' \neq y''$. Then, since $(I - A)$ is non-singular, we must have $c' \neq c''$. We now demonstrate that, as a result, $q'$ and $q''$ must be distinct, where $q'$ and $q''$ are price supports of $y'$ and $y''$ respectively, satisfying conditions (5.5), (5.6) of Lemma 5.5.2.

Suppose $q' = q''$. Then, by using (5.5), we have:

$$\begin{aligned} w(c') - q'c' &\geq w((1/2)(c' + c'')) - q'((1/2)(c' + c'')) \\ &> (1/2)[w(c') - q'c'] + (1/2)[w(c'') - q'c''] \end{aligned}$$

the strict inequality following from the fact that $w$ is strictly concave on $\mathbb{R}^n_{++}$ and $c' >> 0$ and $c'' >> 0$ by Lemma 5.5.1. Thus, we must have:

$$w(c') - q'c' > w(c'') - q'c'' \tag{5.15}$$

Similarly, we get from (5.5),

$$w(c'') - q''c'' > w(c') - q''c' \tag{5.16}$$

Clearly, if $q' = q''$, (5.15) and (5.16) cannot both hold. Thus, $q' \neq q''$. Now, it follows from (5.7) of Lemma 5.5.2 that $v' \neq v''$, since $(\delta I - A)$ is non-singular.

Without loss of generality, suppose that $v'' > v'$. Define $\mu = (v''/v')$; then $\mu > 1$, and by (5.7), we must have $q'' = \mu q'$.

Denoting $q'c'$ by $M'$, we note that $c'$ is the unique solution to:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & q'c \leq M' \\ and & c \geq 0 \end{array}\right\} (P')$$

Similarly, denoting $q''c''$ by $M''$, we note that $c''$ is the unique solution to:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & q''c \leq M'' \\ and & c \geq 0 \end{array}\right\} (P'')$$

Since $q'' = \mu q'$, it follows that $c''$ is the unique solution to:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & q'c \leq q'c'' \\ and & c \geq 0 \end{array}\right\} (P''')$$

We can now split up our analysis into three cases (i) $q'c'' = M'$, (ii) $q'c'' > M'$, (iii) $q'c'' < M'$.

In case (i), problems $(P')$ and $(P''')$ are the same and so $c'$ and $c''$ must both solve $(P')$, implying $c' = c''$, a contradiction.

In case (ii), we must have $c'' > c'$ by normality of $w$. Thus, we obtain $(I-A)y'' >> (I-A)y'$, which implies that $y'' >> y'$, since $(I-A)^{-1} >> 0$. But, then, we get a contradiction by noting from Lemma 5.5.2, $1 = by'' > by' = 1$.

The analysis of case (iii) is analogous to that of case (ii).

Thus, the hypothesis that there are two non-trivial stationary optimal stocks must be false, and the theorem is proved.

### 5.5.3 An Example of Non-uniqueness of SOS

To emphasize the crucial role of normality of the welfare function in the above result, we now present an example, where normality is violated, and there exist two non-trivial stationary optimal stocks. The idea of the example follows the discussion of this issue in Brock (1973) and Brock and Burmeister (1976); however, we are more explicit in our construction, and we ensure that the example of non-uniqueness can be generated by a strictly concave welfare function on consumption vectors.

The technology is described by a $2 \times 2$ matrix $A$ and a two-dimensional vector, $b$, which are specified as follows:

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}; \; b = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

We define the welfare function, $w$, only on the set $C = \{(c_1, c_2) : c_1 \in [0,1], c_2 \in [0,1]\}$, since the technology does not allow for consumption outside this set on any program after the initial time period. A suitable extension of $w$ from the domain $C$ to $\mathbb{R}^2_+$ can be constructed, preserving the key properties of $w$ on

$C$, but this is somewhat tedious, and is not included here. The function, $w$, is defined on $C$ as follows:

$$w(c_1, c_2) = qc_1 - (1/2)rc_1^2 - c_1c_2 + Qc_2 - (1/2)Rc_2^2$$

where $r = 9.8/8 = 1.225$, $R = 9.8/12 = (2/3)r$, $q = 3$ and $Q = 2.41$. A few of the important relations between the parameters may be noted. We have $r < 1.3$, $R < 1$, and $rR = (9.8)^2/96 = 96.04/96 > 1$. Also, $4r + 6R = 9.8 = (49/5)$.

Note that for all $(c_1, c_2) \in C$,

$$w_1(c_1, c_2) = q - rc_1 - c_2 > 0; \; w_2(c_1, c_2) = Q - Rc_2 - c_1 > 0$$

so that $w$ is increasing in each component of the consumption vector and:

$$\begin{aligned} w_{11}(c_1, c_2) &= -r, \; w_{22}(c_1, c_2) = -R \\ w_{12}(c_1, c_2) &= w_{21}(c_1, c_2) = -1 \end{aligned}$$

so that, using $rR > 1$, $w$ is strictly concave on $C$.

The discount factor is specified to be $\delta = 0.9$.

We will show that $y' = (0.5, 0.5)$ and $y'' = (0.6, 0.4)$ are both stationary optimal stocks. These are stationary stocks with corresponding consumption vectors $c' = (0.25, 0.3)$ and $c'' = (0.3, 0.24) = (c_1' + \varepsilon, c_2' - (6/5)\varepsilon)$, where $\varepsilon = 0.05$. They are clearly non-trivial. Further, the corresponding input levels are given by $x' = Ay' = (0.25, 0.2)$ and $x'' = Ay'' = (0.3, 0.16)$. There is full-employment of labor for both stocks, since $by' = by'' = 1$.

To verify that $y'$ is a SOS, we use the dual approach, and define:

$$p' = (q - rc_1' - c_2', Q - Rc_2' - c_1') = (w_1(c_1', c_2'), w_2(c_1', c_2'))$$

Then, $p' >> 0$, and by concavity of $w$ on $C$, we have:

$$w(c') - p'c' \geq w(c) - p'c \;\; for \; all \; c \in C \tag{5.17}$$

Given the definition of $p'$, we see that the relative price $(p_1'/p_2') = (5/4)$. Since this is a crucial fact in our construction, we provide the necessary calculations as follows. We have:

$$(5 - 4r)c_1' + (5R - 4)c_2' = 0.1c_1' + (1/12)c_2' = 0.05$$

and:

$$(5Q - 4q) = 0.05$$

so that:

$$(5 - 4r)c_1' + (5R - 4)c_2' = (5Q - 4q)$$

and by transposing terms:

$$4(q - rc_1' - c_2') = 5(Q - Rc_2' - c_1')$$

Using the fact that $(p_1'/p_2') = (5/4)$, we have:

$$\begin{aligned} p'(\delta I - A) &= [0.4p_1', 0.5p_2'] = p_2'[0.5, 0.5] \\ &= (1/2)p_2'b \end{aligned}$$

Thus, for all $(x, y) \in \Omega$, we have:

$$p'(\delta y - x) \leq p'(\delta I - A)y = (1/2)p_2'by \leq (1/2)p_2' = p'(\delta I - A)y' \qquad (5.18)$$

Using (5.17) and (5.18), it is straightforward to check that $\{y'\}$ is optimal[4] from $y'$.

To verify that $y''$ is a SOS, we define:

$$p'' = (q - rc_1'' - c_2'', Q - Rc_2'' - c_1'') = (w_1(c_1'', c_2''), w_2(c_1'', c_2''))$$

Then, $p'' >> 0$, and by concavity of $w$ on $C$, we have:

$$w(c'') - p''c'' \geq w(c) - p''c \;\; \textit{for all } c \in C \qquad (5.19)$$

Given the definition of $p''$, we see that the relative price $(p_1''/p_2'') = (5/4)$, so that both stationary stocks have price supports, such that the *relative* price is the same. This is important enough to justify providing the necessary calculations. We have:

$$\begin{aligned} (5 - 4r)c_1'' + (5R - 4)c_2'' &= (5 - 4r)c_1' + (5R - 4)c_2' \\ &\quad + (5 - 4r)\varepsilon - (5R - 4)(6/5)\varepsilon \end{aligned}$$

Now, $(5 - 4r) - (5R - 4)(6/5) = -(4r + 6R) + (5 + (24/5)) = 0$ and so:

$$(5 - 4r)c_1'' + (5R - 4)c_2'' = (5 - 4r)c_1' + (5R - 4)c_2' = 0.05$$

Also, as noted above:

$$(5Q - 4q) = 0.05$$

so that:

$$(5 - 4r)c_1'' + (5R - 4)c_2'' = (5Q - 4q)$$

and by transposing terms:

$$4(q - rc_1'' - c_2'') = 5(Q - Rc_2'' - c_1'')$$

Using the fact that $(p_1''/p_2'') = (5/4)$, we have:

$$\begin{aligned} p''(\delta I - A) &= [0.4p_1'', 0.5p_2''] = p_2''[0.5, 0.5] \\ &= (1/2)p_2''b \end{aligned}$$

[4] Strictly speaking, the pair $(y', p')$ has not been shown to constitute a modified golden-rule since (5.18) is only shown to hold on $C$. However, all programs starting from $y'$ must have consumption vectors in *all periods* which belong to $C$, and so the standard *argument* (which is used to show that the stock associated with a modified golden-rule pair is an SOS) still applies.

Thus, for all $(x, y) \in \Omega$, we have:

$$p''(\delta y - x) \leq p''(\delta I - A)y = (1/2)p_2''by \leq (1/2)p_2'' = p''(\delta I - A)y'' \tag{5.20}$$

Using (5.19) and (5.20), it is straightforward to check that $\{y''\}$ is optimal from $y''$.

We can check that normality is violated by $w$. Denote $p'c'$ by $M'$ and $p''c''$ by $M''$. Then, using (5.17), and the strict concavity of $w$ on $C$, we know that the unique solution $c(p', M')$ to the problem:

$$\left.\begin{array}{ll} Max & w(c) \\ subject\ to & p'c \leq M' \\ and & c \in C \end{array}\right\} (P)$$

is given by $c'$. Consequently, $c(p'/\mu, M'/\mu)$ is also given by $c'$, where $\mu = (p_2'/p_2'')$. But, since $p' = \mu p''$, we have $c(p'', M'/\mu) = c'$; also, of course, $c(p'', M'') = c''$. Now,

$$\begin{aligned} p''c'' &= p_2''[(5/4)c_1'' + c_2''] \\ &= (p_2''/p_2')p_2'[(5/4)c_1' + c_2' + (5/4)\varepsilon - (6/5)\varepsilon] \\ &> (p_2''/p_2')p_2'[(5/4)c_1' + c_2'] \\ &= p'c'/\mu \end{aligned}$$

Thus, we have $M'' > M'/\mu$, but $c_2'' < c_2'$, so that normality of $w$ is violated.

## 5.6 Uniqueness of Interior SOS for Smooth Economies

When the economy is smooth (the reduced form utility function is twice continuously differentiable in the interior of the transition possibility set), the methods of differential topology can be used to demonstrate uniqueness of *interior* stationary optimal stocks. This is done by establishing a connection (mathematically, a homotopy) between the set of SOS in the discounted case with the set of SOS in the undiscounted case.

When future utilities are undiscounted, the notion of optimality (defined in terms of some version of the overtaking criterion) is somewhat different from the one described in Section 5.2. However, we can avoid getting into a full discussion of the undiscounted case by first stating a purely mathematical result (Lemma 5.6.1), which helps us to effectively make the same connection as is mentioned in the preceding paragraph.[5]

Lemma 5.6.1 is used in two ways. First, it helps us to provide a link between the analysis of SOS (in the discounted case) in Sections 5.3 and 5.4 of this paper with that offered in this section, which is in terms of stationary solutions to

[5] For some discussion of optimality in the undiscounted case, see the bibliographic remarks in Section 5.7 below.

Ramsey-Euler equations (Proposition 5.6.1). Second, it allows us to examine (see Lemma 5.6.2) the set of stationary solutions to Ramsey-Euler equations in the undiscounted case. [Note that this can be done without discussing the relation between these solutions in the undiscounted case and any notion of optimal programs in the undiscounted case].

Lemma 5.6.2 provides the appropriate result to establish the uniqueness theorem (Theorem 5.6.1) for interior SOS in the discounted case, by using the homotopy invariance theorem and the degree theorem from differential topology.

Since we will be dealing now with "smooth economies", we strengthen assumption (A.5) of Section 5.2 as follows:

(A.5+) *u is* (i) *upper semicontinuous and* (ii) *concave* on $\Omega$. Further, $u$ is twice continuously differentiable in the interior of $\Omega$.

Let us define $\Omega^0 = \{(x, y) \in int\ \Omega : ||x|| \ < \xi\}$, where $\xi$ is given by (A.3). Then $\Omega^0$ is an open and bounded subset of $int\ \Omega$. Further, if $(x, x) \in int\Omega$, then $(x, x) \in \Omega^0$ by (A.3). We denote the set $\{x : (x, x) \in \Omega^0\}$ by $\Lambda$.

We define a function $G$ from $\Lambda \times [0, 1]$ to $\mathbb{R}^n$ by:

$$G(x, \rho) = u_2(x, x) + \rho u_1(x, x) \tag{5.21}$$

In view of (A.5+), the function $G$ is well-defined[6] by (5.21). We denote the Jacobian matrix of $G$, evaluated at $(x, \rho) \in \Lambda \times [0, 1]$, by $J(x, \rho)$, and the determinant of this matrix by $\det J(x, \rho)$. Given $\rho \in [0, 1]$, the set of solutions in $\Lambda$ to the equation $G(x, \rho) = 0$ is denoted by $M(\rho)$.

**Lemma 5.6.1.** *Suppose* $(k, \rho) \in \Lambda \times [0, 1]$ *satisfies:*

$$u_2(k, k) + \rho u_1(k, k) = 0 \tag{5.22}$$

*then there is* $p \in \mathbb{R}^n_+$ *such that:*

$$u(k, k) + p(\rho k - k) \geq u(x, y) + p(\rho y - x) \quad \textit{for all } (x, y) \in \Omega \tag{5.23}$$

*and* $(k, k)$ *solves the maximization problem:*

$$\left.\begin{array}{ll} Max & u(x, y) \\ subject\ to & \rho y - x \geq \rho k - k \\ and & (x, y) \in \Omega \end{array}\right\} \tag{5.24}$$

*Proof.* Define $p = u_1(k, k)$; then $p \in \mathbb{R}^n_+$. Concavity of $u$ implies that for every $(x, y) \in \Omega$,

$$\begin{aligned} u(x, y) - u(k, k) &\leq u_1(k, k)(x - k) + u_2(k, k)(y - k) \\ &= p(x - k) - \rho p(y - k) \end{aligned}$$

[6] The use of $\rho$ rather than $\delta$ here is deliberate. The discount factor, $\delta$, has been restricted to be less than 1 in our description of the basic model in Section 5.2. In contrast, we definitely want $\rho$ to take on the value 1, as well as values less than 1.

the last line following from (5.22). Transposing terms yields (5.23). Clearly, (5.24) follows directly from (5.23).

Using Lemma 5.6.1, we see that interior SOS are equivalent to stationary solutions of Ramsey-Euler equations (in the discounted case).

**Proposition 5.6.1.** *If* $(k, \delta) \in \Lambda \times (0,1)$, *then the following statements are equivalent:*

*(i)* $u_2(k,k) + \rho u_1(k,k) = 0$.

*(ii)* $k$ *is a stationary optimal stock.*

*Proof.* If (i) holds, then we can use Lemma 5.6.1 to obtain $p \in \mathbb{R}^n_+$ such that:

$$u(k,k) + p(\delta k - k) \geq u(x,y) + p(\delta y - x) \quad \textit{for all } (x,y) \in \Omega \tag{5.25}$$

Defining $p(t) = \delta^t p$ for $t \geq 0$, we have for all $t \geq 0$ :

$$\delta^t u(k,k) + p(t+1)k - p(t)k \geq \delta^t u(x,y) + p(t+1)y - p(t)x \quad \textit{for all } (x,y) \in \Omega \tag{5.26}$$

and:

$$\lim_{t \to \infty} p(t)k = 0 \tag{5.27}$$

since $\delta \in (0,1)$. Thus by the standard sufficiency result on price characterization of optimality, $\{k\}$ is optimal from $k$, which establishes (ii).

If (ii) holds, then using the fact that $k \in \Lambda$, we know that $k$ solves the maximization problem:

$$\left.\begin{array}{ll} \textit{Max} & u(k,x) + \delta u(x,k) \\ \textit{subject to} & (k,x) \in \textit{int } \Omega \\ \textit{and} & (x,k) \in \textit{int } \Omega \end{array}\right\} \tag{5.28}$$

Then, we obtain (i) as the necessary first-order condition of the problem (5.28).

To proceed with our analysis, we now impose the condition:

(B.1) There is $(\hat{x}, \hat{x}) \in \Omega^0$, such that $G(\hat{x}, 1) = 0$ and $\det J(\hat{x}, 1) \neq 0$.

**Lemma 5.6.2.** *Under condition (B.1), the equation* $G(x,1) = 0$ *has exactly one solution for* $x \in \Lambda$.

*Proof.* By condition (B.1), $\hat{x} \in \Lambda$ is a solution of the equation $G(x,1) = 0$. Suppose $x' \in \Lambda$ is also a solution to $G(x,1) = 0$, with $x' \neq \hat{x}$.

Using Lemma 5.6.1 for $\rho = 1$, we know that $(\hat{x}, \hat{x})$ and $(x', x')$ are both solutions to:

$$\left.\begin{array}{ll} \textit{Max} & u(x,y) \\ \textit{subject to} & y - x \geq 0 \\ \textit{and} & (x,y) \in \Omega \end{array}\right\} \tag{5.29}$$

By convexity of $\Omega$ and concavity of $u$, we know that $(x(\lambda), x(\lambda)) = \lambda(\hat{x}, \hat{x}) + (1-\lambda)(x', x')$ must also solve (5.29) for all $\lambda \in (0,1)$, and for every $\lambda \in (0,1)$, we have $(x(\lambda), x(\lambda)) \neq (\hat{x}, \hat{x})$. But since $\det J(\hat{x}, 1) \neq 0$, the solution $\hat{x}$ is locally unique, and therefore for $\lambda$ sufficiently close to 1, we get a contradiction. This proves the lemma.

We now impose an additional condition on the set of solutions to $G(x, \rho) = 0$ :

(B.2) There is $\delta \in (0, 1)$, and an open set $C$, such that $\bar{C} \subset \Lambda$, and $M(\rho) \subset C$ for all $\rho \in [\delta, 1]$.

Here $\bar{C}$ is the closure of $C$. The condition implies that for every $\rho \in [\delta, 1]$, the boundary of $C$ contains no solution to $G(x, \rho) = 0$.

In order to keep our exposition self-contained, we state the two results from differential topology that we will need for the main result of this section. These results, and their complete proofs, can be found in Ortega and Rheinboldt (1970, Chapter 6), who follow the approach of Erhard Heinz (1959) in providing an elementary analytic theory of the degree of a mapping[7].

**Homotopy Invariance Theorem:**[Ortega and Rheinboldt (1970, Result 6.2.2, p.156)]

Let $C$ be open and bounded and $H : \bar{C} \times [0, 1] \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$ a continuous map from $\bar{C} \times [0, 1]$ into $\mathbb{R}^n$. Suppose, further, that $H(x, \rho) \neq 0$ for all $(x, \rho) \in \partial C \times [0, 1]$. Then, $\deg(H(\cdot, \rho), C)$ is constant for all $\rho \in [0, 1]$.

**Degree Theorem:** [Ortega and Rheinboldt (1970, Result 6.2.9, p. 159)]

Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on the open set $D$, and $C$ an open, bounded set such that $\bar{C} \subset D$. For $x \in D$, denote the Jacobian matrix of $g$ at $x$ by $J_g(x)$. If $0 \notin g(\partial C) \cup g(S(\bar{C}))$, where $S(\bar{C}) = \{x \in \bar{C} : J_g(x)$ is singular $\}$, either $\{x \in C : g(x) = 0\}$ is empty and $\deg(g, C) = 0$, or $\{x \in C : g(x) = 0\}$ consists of finitely many points $x^1, ..., x^{m,}$ and:

$$\deg(g, C) = \sum_{j=1}^{m} sgn \ \det \ J_g(x^j)$$

where $sgn$ denotes the sign function[8].

We now state and prove the main result of this section. The approach to the uniqueness result may be indicated as follows. Using the Degree theorem, it is possible to evaluate the degree of $G(\cdot, \rho)$ in a simple case, which is when $\rho = 1$ in our context, since there is a unique solution to $G(x, 1) = 0$ on $\Lambda$. To evaluate the degree of $G(\cdot, \rho)$ in the case we are really interested in, namely when $\rho = \delta$, we create a homotopy between the two functions, $G(\cdot, 1)$ and $G(\cdot, \delta)$, and apply the Homotopy Invariance theorem; this is where condition (B.2) is used. This procedure yields one evaluation of the degree of $G(\cdot, \delta)$. However, applying the

[7] That is, in their presentation of degree theory, the results do not involve any concept, not familiar from standard real analysis in Euclidean spaces; nor do their proofs.

[8] That is, $sgn$ is a function from $\mathbb{R}$ to $\{-1, 0, +1\}$, satisfying $sgn(r) = +1$ if $r > 0$, $sgn(r) = -1$ if $r < 0$, and $sgn(r) = 0$ if $r = 0$.

Degree theorem to $G(\cdot, \delta)$, we get another evaluation of the degree of $G(\cdot, \delta)$ in terms of the behavior of the Jacobian of $G(\cdot, \delta)$ at the zeroes of the function. The idea then is to impose a hypothesis restricting this behavior in a way that in turn yields an appropriate restriction on the number of zeroes of the function.

**Theorem 5.6.1.** *Suppose* $\det J(x, \delta) \neq 0$ *for all* $x \in M(\delta)$*, and further the sign of* $\det J(x, \delta)$ *is the same for all* $x \in M(\delta)$*. Then there is only one interior SOS when the discount factor is* $\delta$*.*

*Proof.* Under the hypothesis, we need to show that $M(\delta)$ is a singleton. Define $f : \Lambda \to \mathbb{R}$ by $f(x) = G(x, 1)$. The Degree theorem then gives us a formula for computing the degree of $f$ on $C$, where $C$ is given in condition (B.2). Applying the theorem to $f$ on $C$, we get (in view of Lemma 5.6.2):

$$\deg(f, C) = sgn \det J_f(\hat{x}) \equiv sgn \det J(\hat{x}, 1) \tag{5.30}$$

where $\hat{x}$ is given by condition (B.1). Thus, the $\deg(f, C)$, the degree of $f$ on $C$, is either $+1$ or $-1$.

Define $F : \Lambda \to \mathbb{R}$ by $F(x) = G(x, \delta)$. We now show that $\deg(F, C) = \deg(f, C)$, by establishing a homotopy between $f$ and $F$. To this end, define $H : \bar{C} \times [\delta, 1] \to \mathbb{R}^n$ by:

$$H(x, \rho) = G(x, \rho)$$

and note that $C$ is open and bounded, and $H$ a continuous map from $\bar{C} \times [\delta, 1]$ to $\mathbb{R}^n$. Further, by condition (B.2), $H(x, \rho) \neq 0$ for all $(x, \rho) \in \partial C \times [\delta, 1]$, where $\partial C$ denotes the boundary of $C$. Thus, by the Homotopy Invariance theorem, $\deg(H(\cdot, \rho), C)$ is constant for $\rho \in [\delta, 1]$. In particular, then, $\deg(F, C) = \deg(f, C)$, and so $\deg(F, C)$ is either $+1$ or $-1$.

Now, applying the Degree theorem to $F$ on $C$, we know that $M(\delta)$ consists of finitely many points $x^1, ..., x^m$, and:

$$\deg(F, C) = \sum_{j=1}^{m} sgn \ \det J_F(x^j) \equiv \sum_{j=1}^{m} sgn \ \det J(x^j, \delta) \tag{5.31}$$

The hypothesis of the Theorem ensures that $\det J(x^j, \delta) \neq 0$ for all $x^j$, and further the sign of $\det J(x^j, \delta)$ is the same for all $j = 1, ..., m$. Since we know that $\deg(F, C)$ is either $+1$ or $-1$, (5.31) implies that we must have $m = 1$. Thus, $M(\delta)$ is a singleton, and there is only one interior SOS for the discount factor, $\delta$.

**Remark:**

Brock (1973) showed that if $J(x, \rho)$ is non-singular over $M(\rho)$ for each $\rho \in (\rho_1, 1)$, then $M(\rho)$ is a singleton for each $\rho \in (\rho_1, 1)$. Benhabib and Nishimura (1979) provided conditions, which appear in the above Theorem, under which $J(x, \rho)$ might be singular for some $\rho \in (\delta, 1)$, but $M(\delta)$ is a singleton.

## 5.7 Bibliographic Remarks

**Sections 5.3 and 5.4:**
The approach to existence of stationary optimal stocks that we have followed is a primal one, because it is the most direct one, and it economizes on the assumptions used. However, the dual approach provides, in addition, a supporting price vector, and the quantity-price pair is then referred to as a modified golden-rule. The price support is useful in looking at issues related to uniqueness and global asymptotic stability of stationary optimal stocks. This dual approach is surveyed in Mitra (2005).

We have confined our analysis to the case in which future utilities are discounted. In the undiscounted case, programs are compared by using some version of the overtaking criterion. The approach to the existence of stationary optimal stocks in this context is somewhat different. It does not involve the fixed point argument, which is replaced by arguments based on standard constrained optimization theory. The subsequent step of showing that the *golden-rule stock*, found as a solution to the constrained optimization problem, is indeed optimal among all programs starting from that stock, is more complicated, and makes essential use of duality theory and the price support to the golden-rule stock. The complication arises from the fact that the convenient transversality condition (in the discounted case) is not available in the undiscounted case. The reader is referred especially to the contributions by Brock (1970) and Peleg (1973), which are based on the earlier contributions by Gale (1967) and McKenzie (1968).

The price-supported golden-rule is particularly useful in studying long-run dynamic behavior of optimal programs in the undiscounted case. This has been effectively demonstrated in applications of the theory to study the Faustmann solution in the forest management problem (see Mitra and Wan (1986)) and to analyze the choice of technique in development planning (see Khan and Mitra (2005)).

There is no primal approach to the existence problem in the undiscounted case, corresponding to the one presented here for the discounted case. It is of interest to note that it is the dual approach which is employed by Mitra (1991) in establishing existence of stationary optimal stocks in the undiscounted case in models with a *non-convex* transition possibility set, which satisfies a star-shaped property.

**Section 5.5:**
The approach of this section is based on Brock (1973) and Brock and Burmeister (1976), emphasizing the normality property of the welfare function, based on consumption alone. However, unlike these papers, we emphasize the methods of convex analysis, and refrain from making differentiability assumptions on the welfare function. Stationary optimal stocks turn out to be *not* in the interior of the transition possibility set, making the framework of this section distinctly different from that used in Section 5.6.

Instead of a fixed coefficients Leontief type of technology with no-joint production used in this section, Brock (1973) and Brock and Burmeister (1976) use a non-linear activity analysis model, and appeal to the non-substitution theorem. We have presented the results in the more restrictive framework, because the arguments involved are very transparent in this case. Some of this theory can even be generalized to settings with joint production, provided an approppriate version of the non-substitution theorem holds in that framework; for this theory, see Benhabib and Nishimura (1979).

**Section 5.6:**
The methods of differential topology were used to address uniqueness problems in general equilibrium theory by Dierker (1972). They were then used in optimal growth models by Brock (1973) and Benhabib and Nishimura (1979).

We have presented this theory so that a reader, familiar only with standard concepts in real analysis, should be able to follow the results without any difficulty. Specifically, concepts and terminology used in differential topology have been avoided.

For smooth economies, it is possible to develop a connection between the normality assumption in Section 5.5, and the hypothesis on the behavior of the Jacobian at the zeroes of the relevant function used in Section 5.6. This is explored in detail in Benhabib and Nishimura (1979).

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